Worldline Approach of Topological BF Theory

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Abstract

We present a worldline description of topological non-abelian BF theory in arbitrary space-time dimensions. It is shown that starting with a trivial classical action defined on the worldline, the BRST cohomology has a natural representation as the sum of the de Rham cohomology. Based on this observation, we construct a second-quantized action of the BF theory. Interestingly enough, this theory naturally gives us a minimal solution to the Batalin-Vilkovisky master equation of the BF theory. Our formalism sheds some light not only on an interplay between the Witten-type and the Schwarz-type topological quantum field theories but also on the role of the Batalin-Vilkovisky antifields and ghosts as geometrical and elementary objects.

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1 Introduction

In recent years, we have seen an interesting progress on a covariant quantization of Green-Schwarz superstring theories, which was a long-standing problem for about twenty years since appearance of a paper by Green and Schwarz [1], by using pure spinors [2]. (See also related papers [3].) The formalism depends on a very simple form of the BRST charge $Q_B = \oint \lambda^\alpha d_\alpha$, where λ^α is a pure spinor satisfying the pure spinor equation $\lambda^\alpha \Gamma^a_{\alpha\beta} \lambda^\beta = 0$ and d_α is a spinorial covariant derivative.

In a lecture note introducing the pure spinor formalism of superparticle and superstring [4], Berkovits has discussed that the 10D super Yang-Mills theory can be obtained through the BRST quantization of a superparticle action involving pure spinors, just as the 3D Chern-Simons theory which is in essence a topological theory, can be obtained via the BRST quantization of a particle action [5]. This worldline description of the 3D Chern-Simons theory was gained by dimensionally reducing a worldsheet action for a Chern-Simons string theory by Witten [6]³ to a worldline action and was introduced as just a prelude to the pure spinor formalism, but the worldline description is of interest in its own right from some reasons mentioned below.

For instance, it is nowadays well known that there are two types of topological quantum field theories. One is called the Witten-type topological quantum field theories, or the cohomological type where the classical action is some topological invariants or simply zero [8, 9]. This type of topological quantum field theories was originally introduced to understand Donaldson invariants defined on 4D differentiable manifolds. The other is sometimes called the Schwarz-type topological quantum field theories, for which the classical action has the form of Chern-Simons action or BF action [10, 11, 12]. This type of topological quantum field theories was originally developed not only to make possible the quantization of linear p-form theory but also to formulate the Ray-Singer analytical torsions of the de Rham complex in a field theoretic language [13]. These two types of topological quantum field theories share the property that their partition functions are independent of the metric and that the only observables are topological invariants of the underlying space-time manifold, but appear to be disconnected as a field theory since the Schwarz-type has a nontrivial classical action while the Witten-type has a trivial action at least classically and therefore possesses a well-known topological symmetry. The worldline approach which we wish to investigate in this paper bridges the gap between the two types of topological field theories to some extent, which is one reason behind the motivations of the present paper.

Another interesting reason of the worldline approach is that we can construct a second-quantized theory of the Schwarz-type topological quantum field theories by using the BRST charge of the worldline action in a natural way. This second-quantized action includes the Batalin-Vilkovisky antifields in additon to a tower of ghosts and is found to be a minimal solution to the BV master equation, which is a heart of the Batalin-Vilkovisky quantization algorithm [14]. Thus, the formalism at hand gives us a different standpoint of the Batalin-Vilkovisky algorithm and antifields, and also provides us a geometrical origin of the antifields.

In section 2, from the viewpoint of the canonical quantization, we analyze the BRST coho-

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³A supersymmetric extension was considered in [7].

mology of the worldline approach in detail in order to explain why its BRST cohomology can describe the moduli space of the Schwarz-type topological quantum field theories. In section 3, we construct a second-quantized theory corresponding to BF theory. Section 4 is devoted to discussions.

2 Review of the Worldline Approach

To begin with, we shall investigate the worldline approach for topological field theories developed by Berkovits [4] in some detail from a slightly different perspective. The theory involves $D = n + 2$ bosonic variables x^μ representing the position of a particle and their canonical conjugate momenta P_μ in addition to the Lagrange multipliers l^μ imposing the constraints $P_\mu \approx 0$. In this article, we limit ourselves to consider the cases $D \geq 2$, i.e., $n \geq 0$. The classical action of the system is of form

$$S_c = \int d\tau L_c = \int d\tau (\dot{x}^\mu P_\mu + l^\mu P_\mu), \tag{1}$$

where $\dot{x}^\mu = \frac{\partial x^\mu}{\partial \tau}$.

Here let us quantize the system in a canonical manner. The canonical conjugate momenta for x^μ and l^μ are given by

$$
\begin{aligned}
P_\mu &= \frac{\delta S_c}{\delta \dot{x}^\mu}, \\
\pi_\mu &= \frac{\delta S_c}{\delta \dot{l}^\mu} \approx 0.
\end{aligned}
\tag{2}
$$

The latter equation gives rise to the primary constraints. The Hamiltonian H is then defined as

$$H = P_\mu \dot{x}^\mu + \pi_\mu \dot{l}^\mu - L_c = -l^\mu P_\mu. \tag{3}$$

From this Hamiltonian and the primary constraints, one obtains the secondary constraints

$$P_\mu \approx 0. \tag{4}$$

Using the secondary constraints, the Hamiltonian H is weakly zero, so we have no more constraints. At the same time, the secondary constraints make the classical action vanish, thereby implying a topological nature of the classical system[4], that is, this system is an instance of the Witten-type topological field theories. It turns out that the primary and secondary constraints constitute the first-class constraints whose generator is given by

$$G = \int d\tau (-\dot{\varepsilon}_\mu \pi^\mu + \varepsilon_\mu P^\mu). \tag{5}$$

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[4]This can be also seen more directly by performing the path integral over l^μ in the action (1) and then over P_μ.

Actually the generator yields the topological symmetry

$$
\begin{aligned}
\delta x^\mu &= -\varepsilon^\mu, \\
\delta l^\mu &= \dot{\varepsilon}^\mu, \\
\delta P_\mu &= 0,
\end{aligned}
\tag{6}
$$

under which the classical action (1) is manifestly invariant.

Now let us move on to the BRST quantization. The topological symmetry gives rise to the following BRST transformation:

$$
\begin{aligned}
\delta_B x^\mu &= -c^\mu, \\
\delta_B l^\mu &= \dot{c}^\mu, \\
\delta_B P_\mu &= 0, \\
\delta_B b_\mu &= B_\mu, \\
\delta_B c^\mu &= \delta_B B_\mu = 0.
\end{aligned}
\tag{7}
$$

Since we adopt the gauge conditions for the topological symmetry

$$
l^\mu + \frac{1}{2} P^\mu = 0,
\tag{8}
$$

the gauge-fixed, BRST-invariant action is given by

$$
\begin{aligned}
S &= \int d\tau \Big[\dot{x}^\mu P_\mu + l^\mu P_\mu + \delta_B \Big(b_\mu (l^\mu + \frac{1}{2} P^\mu) \Big) \Big], \\
&= \int d\tau \Big[\dot{x}^\mu P_\mu + l^\mu P_\mu + B_\mu (l^\mu + \frac{1}{2} P^\mu) + \dot{c}^\mu b_\mu \Big].
\end{aligned}
\tag{9}
$$

In order to simplify this quantum action further, we shall carry out the path integral over the auxiliary fields l^μ and B_μ whose result is given by

$$
S = \int d\tau \Big[\dot{x}^\mu P_\mu - \frac{1}{2} P_\mu P^\mu + \dot{c}^\mu b_\mu \Big].
\tag{10}
$$

The Noether theorem makes it possible to construct the BRST charge

$$
Q_B = i c^\mu P_\mu.
\tag{11}
$$

The action (10) is also invariant under the scale transformation

$$
\begin{aligned}
c^\mu &\rightarrow e^\rho c^\mu, \\
b_\mu &\rightarrow e^{-\rho} b_\mu,
\end{aligned}
\tag{12}
$$

where ρ is a real parameter, so we can define the ghost number charge through the Noether theorem by

$$
Q_c = -i c^\mu b_\mu.
\tag{13}
$$

3

Since we have the commutators

$$[Q_c, c^\mu] = c^\mu,$$
$$[Q_c, b_\mu] = -b_\mu, \tag{14}$$

c^μ and b_μ have respectively $+1$ and -1 ghost numbers. In the above, the (anti-)commutation relations are set up as usual by

$$[x^\mu, P_\nu] = i\delta^\mu_\nu,$$
$$\{c^\mu, b_\nu\} = i\delta^\mu_\nu, \tag{15}$$

with the other (anti-)commutators vanishing.

The quantization is incomplete unless one fixes the concrete representation of the above algebra. At this stage, we find that there is a natural representation which is nothing but the representation on the space of differential forms.

$$\mathcal{H} = \Omega(M) = \bigoplus_{p=0}^{D} \Omega^p(M), \tag{16}$$

where M and $\Omega^p(M)$ denote the D-dimensional manifold and the space of the p-forms, respectively. On this space, the variables in the worldline approach are represented as operators and have the following correspondence with differential forms:

$$x^\mu \longleftrightarrow x^\mu \otimes,$$
$$P_\mu \longleftrightarrow -i\frac{\partial}{\partial x^\mu},$$
$$c^\mu \longleftrightarrow dx^\mu \wedge,$$
$$b_\mu \longleftrightarrow i \times i_{\frac{\partial}{\partial dx^\mu}}, \tag{17}$$

where i_V means the interior product which is an operation producing $(k-1)$-form from k-form by contracting the differential form with the vector field V. Then, the physical Hilbert space, which is denoted as $|\psi>$, should be annihilated by the BRST charge

$$Q_B|\psi >= 0. \tag{18}$$

In this representation, we also have the following correspondence:

$$|0> \longleftrightarrow 1,$$
$$c^\mu|0> \longleftrightarrow dx^\mu,$$
$$\cdots \longleftrightarrow \cdots,$$
$$c^{\mu_1} \cdots c^{\mu_D}|0> \longleftrightarrow dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_D},$$
$$Q_B \longleftrightarrow dx^\mu \wedge \frac{\partial}{\partial x^\mu} = d,$$
$$-H \longleftrightarrow \frac{1}{2}\nabla, \tag{19}$$

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where $\triangledown$ is the Laplacian operator. Here we have denoted by $|0>$ the vector annihilated by Q_c in addition to Q_B, so $|0>$ belongs to the ghost number 0 sector.[5] Since $[Q_c, c^\mu] = c^\mu$ from (14), the ghost number of a general state $c^{\mu_1} \cdots c^{\mu_p}|0>$ is identified with the form-degree p. Also notice that the physical state condition leads to the zero energy condition, so the whole physical state is consisted of only the ground states, which are simply the harmonic forms. In other words, the BRST cohomology $\mathcal{H}$ at hand is the direct sum of the de Rham cohomology group $H^p(M)$ of forms on the D-dimensional manifold M:

$$\mathcal{H} = \bigoplus_{p=0}^{D} H^p(M). \tag{20}$$

Here recall that the space $\mathcal{N}$ of classical solutions of the $3D$ Chern-Simons theory (and the BF theory in arbitrary dimensions) is a finite dimensional de Rham cohomology group (and the direct sum of two de Rham cohomology groups) [10]. Hence, it is natural to expect that the moduli space of the Schwarz-type of topological field theories might be described in terms of the BRST cohomology of the worldline approach mentioned above, which is of the Witten-type [8, 9]. In fact, Berkovits has shown that this is indeed the case for the $3D$ Chern-Simons theory [4]. One of motivations behind the present article is show explicitly that this holds for the BF theory [10, 11, 12] in arbitrary dimensions as well. It is worth noting that the BF theory in more than three space-time dimensions has an on-shell reducible symmetry in addition to the usual Yang-Mills gauge symmetry, so our application of the worldline approach to the BF theory is not so obvious as we consider naively.

To close this section, it is valuable to point out that the b ghost associated to the τ-reparametrization symmetry can be found as follows: The Hamiltonian H is rewritten as

$$H = -l^\mu P_\mu = \frac{1}{2} P_\mu^2. \tag{21}$$

The fundamental equation which the b ghost must satisfy is [6]

$$\{Q_B, b\} = H. \tag{22}$$

Hence, we can make the b ghost by

$$b = -\frac{1}{2} b_\mu P^\mu. \tag{23}$$

This form of the b ghost is physically reasonable since the reparametrization is part of more huge topological symmetry. Note that this b ghost is a composite field as that in the pure spinor formalism.

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[5]As an alternative interpretation of this representation, one could regard b_μ as annihilation operators and c^μ as creation operators. Then, the whole physical state is constructed by operating a finite number of creation operators c^μ on the "vacuum" $|0>$ which is destroyed by annihilation operators b_μ.

3 Worldline Description of BF theory

In this section, on the basis of the observation in previous section, we wish to present a worldline description of BF theory in a general space-time dimension. Since the path of argument is similar in both the abelian and the non-abelian gauge groups, we shall discuss only the case of the non-abelian gauge group, from which we can extract the abelian BF theory in a straightforward way.

The classical action of the BF theory in $D = n + 2 \geq 2$ space-time dimensions takes the form[6]

$$S = \int_{M_D} Tr(BF), \tag{24}$$

where A indicates a Lie algebra valued 1-form connection and F is its curvature 2-form defined by $F = dA + A^2$, and B is a section belonging to $\Omega^n(M)$, i.e., a Lie algebra valued n-form. The equations of motion from this action read

$$\begin{aligned} F &= 0, \\ DB &= 0, \end{aligned} \tag{25}$$

where the covariant derivative is defined as $D = d + [A, \ ]$. The action is invariant under the conventional Yang-Mills gauge transformation with the 0-form gauge parameter $\varepsilon(x)$ and the non-abelian symmetry associated with B field with the $(n-1)$-form gauge parameter $\lambda(x)$

$$\begin{aligned} \delta A &= D\varepsilon, \\ \delta B &= [B, \varepsilon] + D\lambda. \end{aligned} \tag{26}$$

When one attempts to quantize this system, a well-known complication appears owing to the latter transformation in four and higher space-time dimensions. Namely, the non-abelian symmetry for B field is on-shell reducible in the sense that $\lambda = D\lambda'$ with λ' being any $(n-2)$-form becomes the zero modes from the equation of motion $F = 0$. (Note that such a sequence of reducible symmetries exists until λ' is a 0-form.) In order to quantize such the on-shell reducible theory, one might rely on the Batalin-Vilkovisky algorithm [12].

Now we are ready to present a worldline description of the above non-abelian BF theory in $D = n + 2$ space-time dimensions. Before doing that, let us first recall the result in the previous section that the BRST cohomology of the worldline approach is $\mathcal{H} = \bigoplus_{p=0}^{n+2} H^p(M)$ whereas the moduli space of the BF theory is $\mathcal{N} = H^n(M) \oplus H^1(M)$ [10].[7] Thus, if we want to make $\mathcal{N}$ coincide with $\mathcal{H}$, it is necessary to add the missing de Rham cohomology groups to $\mathcal{N}$. Actually, such the cohomology groups are precisely supplied by a tower of reducible ghosts and the Yang-Mills ghost as well as the Batalin-Vilkovisky antifields as we will show shortly.

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[6]The wedge product among forms is always understood.

[7]Precisely speaking, this is true only when the gauge group is the abelian group. In the present case, we have a non-abelian generalization of it since there is the non-linear nilpotent operator D from the equations of motion $F = 0$.

Next, corresponding to A and B fields in the BF theory, let us introduce two kinds of functionals $\Psi(c,x)$ with ghost number 1 and $\Phi(c,x)$ with ghost number n, and then expand them in the powers of c^μ which are the topological ghosts in the worldline action of a particle

$$
\begin{aligned}
\Psi(c,x) &= C(x) + c^\mu A_\mu(x) + \frac{1}{2}c^{\mu_1}c^{\mu_2}B^*_{\mu_1\mu_2}(x) + \cdots + \frac{1}{(n+2)!}c^{\mu_1}\cdots c^{\mu_{n+2}}B^*_{\mu_1\cdots\mu_{n+2}}(x), \\
\Phi(c,x) &= B(x) + c^\mu B_\mu(x) + \frac{1}{2}c^{\mu_1}c^{\mu_2}B_{\mu_1\mu_2}(x) + \cdots + \frac{1}{n!}c^{\mu_1}\cdots c^{\mu_n}B_{\mu_1\cdots\mu_n}(x) \\
&+ \frac{1}{(n+1)!}c^{\mu_1}\cdots c^{\mu_{n+1}}A^*_{\mu_1\cdots\mu_{n+1}}(x) + \frac{1}{(n+2)!}c^{\mu_1}\cdots c^{\mu_{n+2}}C^*_{\mu_1\cdots\mu_{n+2}},
\end{aligned}
\tag{27}
$$

where the expanded terms terminate at a finite stage since c^μ are anticommuting and the space-time dimension is now $D = n+2$. Then, we can propose the following second-quantized action which is a natural generalization of the BF theory:

$$
S = \int_{M_D} d^D x\, d^D c\; Tr\Phi(Q_B\Psi + \Psi\Psi),
\tag{28}
$$

where Q_B is the BRST charge of the worldline theory. Here the ghost measure is defined as $\int d^D c\, c^1\cdots c^D = 1$. This action is invariant under the gauge transformations

$$
\begin{aligned}
\delta\Psi &= Q_B\Omega + [\Psi,\Omega], \\
\delta\Phi &= [\Phi,\Omega] + Q_B\Lambda + [\Psi,\Lambda],
\end{aligned}
\tag{29}
$$

with Ω and Λ having 0 and $n-1$ ghost numbers, respectively. Henceforth, we shall use the following notation: we define the total degree of arbitrary forms by the sum of the form degree and ghost number. The grading is then determined by the total degree and the square bracket means the commutator or anti-commutator depending on the grading. Concretely, we have the definition $[X,Y] = XY - (-)^{xy}YX$ for a form X with the total degree x and a form Y with the total degree y. The topological ghosts c^μ are assumed to have the total degree 1. Moreover, we define a general p-form by $A = \frac{1}{p!}dx^{\mu_1}\cdots dx^{\mu_p}A_{\mu_1\cdots\mu_p}$, instead of the conventional definition $A = \frac{1}{p!}A_{\mu_1\cdots\mu_p}dx^{\mu_1}\cdots dx^{\mu_p}$, since our definition is consistent with the expansion (27). Otherwise, we would have numerous ugly factors of signature in the action and the gauge transformations in components.

The action (28) leads to the equations of motion

$$
\begin{aligned}
Q_B\Psi + \Psi\Psi &= 0, \\
Q_B\Phi + [\Psi,\Phi] &= 0.
\end{aligned}
\tag{30}
$$

Substituting the functionals (27) into the action (28), we have an action for each component field by integrating over the topological ghosts

$$
\begin{aligned}
S &= \int_{M_D} Tr\Big[B_{(n,0)}\big(F_{(2,0)} + [C_{(0,1)}, B^*_{(2,-1)}]\big) + A^*_{(n+1,-1)}DC_{(0,1)} \\
&+ C^*_{(n+2,-2)}C_{(0,1)}C_{(0,1)} + \sum_{p=1}^{n} B_{(n-p,p)}\big(DB^*_{(p+1,-p)} + [C_{(0,1)}, B^*_{(p+2,-p-1)}] \\
&+ \frac{1}{2}\sum_{p'=0}^{p-2}[B^*_{(p'+2,-p'-1)}, B^*_{(p-p',-p+p'+1)}]\big)\Big],
\end{aligned}
\tag{31}
$$

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where we have put subscripts on fields in order to indicate the form-degree and ghost number such that $X_{(p,q)}$ in general implies the field X with p-form and ghost number q.

Moreover, provided that we also expand the gauge parameter functionals Ω and Λ in terms of c^μ as

$$\begin{aligned}
\Omega(c,x) &= \omega(x) + c^\mu \omega_\mu(x) + \frac{1}{2} c^{\mu_1} c^{\mu_2} \omega_{\mu_1 \mu_2}(x) + \cdots + \frac{1}{(n+2)!} c^{\mu_1} \cdots c^{\mu_{n+2}} \omega_{\mu_1 \cdots \mu_{n+2}}(x), \\
\Lambda(c,x) &= \lambda(x) + c^\mu \lambda_\mu(x) + \frac{1}{2} c^{\mu_1} c^{\mu_2} \lambda_{\mu_1 \mu_2}(x) + \cdots + \frac{1}{(n+2)!} c^{\mu_1} \cdots c^{\mu_{n+2}} \lambda_{\mu_1 \cdots \mu_{n+2}},
\end{aligned} \qquad (32)$$

then the gauge transformations are explicitly written out in components (for $2 \leq p \leq n+2$ and $1 \leq q \leq n$)

$$\begin{aligned}
\delta C_{(0,1)} &= [C_{(0,1)}, \omega_{(0,0)}], \\
\delta A_{(1,0)} &= D\omega_{(0,0)} + [C_{(0,1)}, \omega_{(1,-1)}], \\
\delta B^*_{(p,-p+1)} &= D\omega_{(p-1,-p+1)} + [C_{(0,1)}, \omega_{(p,-p)}] + \sum_{p'=2}^{p} [B^*_{(p-p'+2,-p+p'-1)}, \omega_{(p'-2,-p'+2)}], \\
\delta B_{(0,n)} &= [B_{(0,n)}, \omega_{(0,0)}] + [C_{(0,1)}, \lambda_{(0,n-1)}], \\
\delta B_{(q,n-q)} &= D\lambda_{(q-1,n-q)} + [C_{(0,1)}, \lambda_{(q,n-q-1)}] + \sum_{q'=0}^{q} [B_{(q-q',n-q+q')}, \omega_{(q',-q')}]
\end{aligned}$$

$$\begin{aligned}
&+ \sum_{q'=0}^{q-2} [B^*_{(q-q',-q+q'+1)}, \lambda_{(q',n-q'-1)}], \\
\delta A^*_{(n+1,-1)} &= D\lambda_{(n,-1)} + [C_{(0,1)}, \lambda_{(n+1,-2)}] + [A^*_{(n+1,-1)}, \omega_{(0,0)}] \\
&+ \sum_{p'=0}^{n} [B_{(n-p',p')}, \omega_{(p'+1,-p'-1)}] + \sum_{p'=0}^{n-1} [B^*_{(n-p'+1,-n+p')}, \lambda_{(p',n-p'-1)}], \\
\delta C^*_{(n+2,-2)} &= D\lambda_{(n+1,-2)} + [C_{(0,1)}, \lambda_{(n+2,-3)}] + [A^*_{(n+1,-1)}, \omega_{(1,-1)}] + [C^*_{(n+2,-2)}, \omega_{(0,0)}] \\
&+ \sum_{p'=0}^{n} [B_{(n-p',p')}, \omega_{(p'+2,-p'-2)}] + \sum_{p'=0}^{n} [B^*_{(n-p'+2,-n+p'-1)}, \lambda_{(p',n-p'-1)}]. \qquad (33)
\end{aligned}$$

A few comments are in order. One important comment is that the action (28), or equivalently (31), turns out to be a minimal solution to the Batalin-Vilkovisky master equation [14]. This is easily checked by a string field theoretic technique where using the antibracket, the Batalin-Vilkovisky master equation is given by

$$\begin{aligned}
(S,S) &\equiv \int d^D x \, d^D c \, Tr \frac{\delta S}{\delta \Psi} \frac{\delta S}{\delta \Phi} \\
&= \int d^D x \, d^D c \, Tr (Q_B \Phi + [\Psi, \Phi])(Q_B \Psi + \Psi \Psi) \\
&= \int d^D x \, d^D c \, Tr \left(Q_B (\Psi Q_B \Phi + \Phi \Psi^2) + [\Psi, \Phi] \Psi^2 \right) \\
&= 0, \qquad (34)
\end{aligned}$$

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where we have used the fact that $\int Q_B(\cdots) = 0$ and Ψ is anticommuting. We can rewrite this master equation to more familiar form for each component field as follows:

$$
\begin{aligned}
(S, S) &\equiv \int d^D x \, Tr\Big(\frac{\partial^l S}{\partial C^*_{(0,-2)}} \frac{\partial^r S}{\partial C_{(0,1)}} + \frac{\partial^l S}{\partial A^*_{(1,-1)}} \frac{\partial^r S}{\partial A_{(1,0)}} + \sum_{p=0}^{n} \frac{\partial^l S}{\partial B^*_{(p,p-n-1)}} \frac{\partial^r S}{\partial B_{(p,-p+n)}}\Big) \\
&= 0,
\end{aligned}
\tag{35}
$$

where we have considered the dual fields for the Batalin-Vilkovisky antifields.

The second comment is that the second-quantized action (28) is automatically equipped with ghosts, ghosts of ghosts and antifields for reducible gauge symmetries for B field in addition to the Yang-Mills ghost and antifield. This in turn implies that we should take account of the Batalin-Vilkovisky antifields on an equal footing with ghosts in order to realize the huge second-quantized symmetry (29) or (33). Of course, it is easy to recover the original BF action (24) with the gauge transformations (26) from the second-quantized action (31) through elimination of ghosts and the antifields by using the huge symmetry (33) at least locally, but not globally. In the Batalin-Vilkovisky algorithm for quantization, recall that the action is a generator for the BRST transformation in the antibracket in the sense that $sX = (X, S)$ and the antifields must be gauge-fixed by selecting a suitable gauge fermion. Actually, using the equation $sX = (X, S)$, it is easy to derive the BRST transformation whose result is given by (for $3 \leq p \leq n + 2$ and $1 \leq q \leq n$)

$$
\begin{aligned}
sC_{(0,1)} &= C_{(0,1)}C_{(0,1)}, \\
sA_{(1,0)} &= DC_{(0,1)}, \\
sB^*_{(2,-1)} &= -(F_{(2,0)} + [C_{(0,1)}, B^*_{(2,-1)}]), \\
sB^*_{(p,-p+1)} &= -(DB^*_{(p-1,-p+2)} + [C_{(0,1)}, B^*_{(p,-p+1)}] + \frac{1}{2}\sum_{p'=0}^{p-4}[B^*_{(p'+2,-p'-1)}, B^*_{(p-p'-2,-p+p'+3)}]), \\
sB_{(0,n)} &= [C_{(0,1)}, B_{(0,n)}], \\
sB_{(q,n-q)} &= DB_{(q-1,n-q+1)} + [C_{(0,1)}, B_{(q,n-q)}] + \sum_{q'=0}^{q-2}[B^*_{(q-q',-q+q'+1)}, B_{(q',n-q')}], \\
sA^*_{(n+1,-1)} &= -(DB_{(n,0)} + [C_{(0,1)}, A^*_{(n+1,-1)}] + \sum_{p'=1}^{n}[B^*_{(p'+1,-p')}, B_{(n-p',p')}]), \\
sC^*_{(n+2,-2)} &= -(DA^*_{(n+1,-1)} + [C_{(0,1)}, C^*_{(n+2,-2)}] + \sum_{p'=0}^{n}[B^*_{(p'+2,-p'-1)}, B_{(n-p',p')}]).
\end{aligned}
\tag{36}
$$

The final remark is related to the extended differential calculus on the universal bundle where the extended differential operator is the sum of the exterior derivative d and the BRST transformation s [9]

$$
\tilde{d} = d + s,
\tag{37}
$$

and the universal 1-form connection $\tilde{A}$ is the sum of the gauge connection A and the Yang-Mills ghost C

$$
\tilde{A} = A_{(1,0)} + C_{(0,1)},
\tag{38}
$$

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since each object on the RHS carries the same total degree 1. It is true that this extended formalism yields the desired BRST transformation very nicely.[8] The use of the total degree also seems to suggest that we could formulate the present theory in terms of the extended differential calculus on the universal bundle. However, after some efforts we have found it difficult to formulate a second-quantized theory in a covariant manner in the framework of the extended differential calculus though we need more study to clarify this point in future.

4 Discussions

In this article, we have presented a worldline description of topological BF theory in arbitrary space-time dimensions and found that this formulation provides a useful tool for obtaining a minimal solution to the Batalin-Vilkovisky master equation without solving it directly, which is usually a tough work especially for the system with reducible on-shell symmetries.[9] In the second-quantized formalism, the ghosts and ghosts of ghost as well as the corresponding antifields are naturally required to participate in the action to realize the gauge symmetry off-shell. In this sense, the antifields play the same role as the ghosts and should be regarded as the geometrical and fundamental objects in the construction of a second-quantized theory. It is remarkable to notice that the missing de Rham cohomology groups in the original BF theory are neatly provided with such the fields by taking account of general functionals.

It is natural to ask whether the present formulation can apply to the other systems to get a minimal solution to the master equation. In this context, we should pay attention to the form of the BRST charge $Q_B = ic^\mu P_\mu = c^\mu \partial_\mu$ and the relation (22). If we introduce the dual BRST charge by $\tilde{Q}_B = -b_\mu P^\mu$, we can have a suggestive equation $\frac{1}{2}\{Q_B, \tilde{Q}_B\} = H$ and as a result obtain the second-order Laplace operator. However, it seems to be difficult to get a nilpotent operator associated to the Laplacian because of the characteristic feature of the topological ghosts being space-time vectors, so the application of the present formulation might be limited to only the system with the first-order differential operator in the kinetic term in the action.

Finally, it is known that when we specify the space-time dimensions to three and the gauge group to the $SO(1,2)$ group, the BF theory is reduced to three-dimensional gravity, which is essentially topological and 1-loop exact [18]. Thus, the BF theory at hand might be relevant to a topological gravity in three dimensions or a closed string sector of superstring theory though we need much works to be done in future to render this idea realistic.

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[8]Here the "flatness" condition for the extended curvature 2-form [15] plays an important role.

[9]See also related works [16, 17].

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